On Predicting Forming Limits Using Hill's Yield Criteria

Siguang Xu and Klaus J. Weinmann

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The analysis of localized necking is strongly dependent on the yield function. To predict forming limits, therefore, numerous yield criteria have been postulated to characterize the plastic deformation of sheet materials. Among them Hill's 1948 and the fourth form of 1979 yield criteria are the most commonly used. A new yield criterion was proposed by Hill in 1993. It uses five independent and easily obtainable material parameters, which makes it flexible in representing the shape of the yield locus for different materials. The present investigation compares these three yield criteria in forming limit predictions based on both the Marciniak and Kuczynski (M-K) approach and the bifurcation analysis. It is observed that the M-K analysis based on Hill's 1993 yield criterion provides forming limit predictions in agreement with experimental data. The bifurcation analysis based on Hill's 1948 yield criterion also provides an acceptable prediction of forming limits for aluminum, although they are slightly higher. All three yield criteria are found to provide acceptable predictions for aluminum-killed (AK) steel based on the M-K method. For brass, only the prediction based on the M-K method and Hill's 1993 yield criterion is close to the trend of experimental data.

Keywords plastic deformations, forming limits, aluminum-killed steel, aluminum sheet

1. Introduction

One of the common failure modes in sheet forming is localized necking. Forming limit diagrams (FLDs) are usually used to characterize the formability of sheet metal. Experimental evidence has shown that material properties, sheet thickness, strain paths, and surface finish are the major factors controlling the formability of sheet metals. Marciniak and Kuczynski (M-K)[1] introduced the concept of an initial imperfection in the sheet, which develops into a localized neck when the load applied to the uniform region of the sheet increases and force equilibrium between the groove and the outside region is maintained. The M-K method has been widely used to predict forming limits of various materials.[2-5] Another approach to the analysis of localized necking considers that a bifurcation mode is assumed to indicate the initialization of localized necking,[6,7] where analysis is based on deformation theory of rigid-plastic material.

The geometric configuration of the yield surface has a significant influence on predicted forming limit strains. Many yield criteria have been proposed to reflect the material properties of sheet metals.[8-14] Hill's 1948 yield criterion has been used extensively to predict forming limits of aluminum-killed (AK) steel based on the M-K method.[15–17] However, for aluminum sheet, significant discrepancies exist between experimental data and predictions when this criterion is used. To accommodate the anomalous behavior of aluminum,[18] a second yield criterion was postulated by Hill in 1979.[13] Analysis based on the fourth form of this yield criterion shows an improvement in forming limit predictions for aluminum sheet.[4] In 1993, Hill proposed a new and user-friendly yield criterion.[14] This criterion has five independent material properties. Thus, it may have flexibility in representing the yield locus of various materials. Since Hill's 1948 and 1979 yield criteria are the most commonly used yield criteria in predicting forming limits, and Hill's 1993 yield criterion has a potential of wide applications, the question may be which yield criterion should be used under certain circumstances. Therefore, the present investigation is focused on the comparison of the yield criteria proposed by Hill in 1948, 1979, and 1993 and their effect on forming limit predictions. The shape of yield loci of these criteria is discussed. The forming limits predicted using both the M-K approach and the bifurcation analysis are compared for the three yield functions. For selected materials, predicted forming limits are compared with experimental data.

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2. Hill's Yield Criteria

2.1 The 1948 Yield Criterion

The original form of this yield function was given as [12]

$$2f = F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_y - \sigma_z)^2 + 2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 = 1 \qquad \text{(Eq 1)}$$

where F, G, H, L, M, and N are constants, which describe the characteristics of material anisotropy. These constants can be determined by tensile yield stresses in the principal anisotropic directions and yield stresses in shear. They can also be determined by introducing strain ratios r_0, r_{45}, and r_{90} for the plane stress condition. Considering sheet metal forming and assuming that the sheet has planar isotropy, Eq 1 reduces to

$$\sigma_1^2 + \sigma_2^2 + r(\sigma_1 + \sigma_2)^2 = (1+r)\sigma_u^2 \qquad \text{(Eq 2)}$$

where σ_u is the in-plane uniaxial tensile stress, r is the normal anisotropic strain ratio, and σ_1 and σ_2 are principal stresses. The loci of Eq 2 are shown in Fig. 1. They are ellipses with major and minor axes depending on the r value. A higher value of r causes a higher value of the yield stress under biaxial tension. For balanced biaxial tension, Eq 2 reduces to

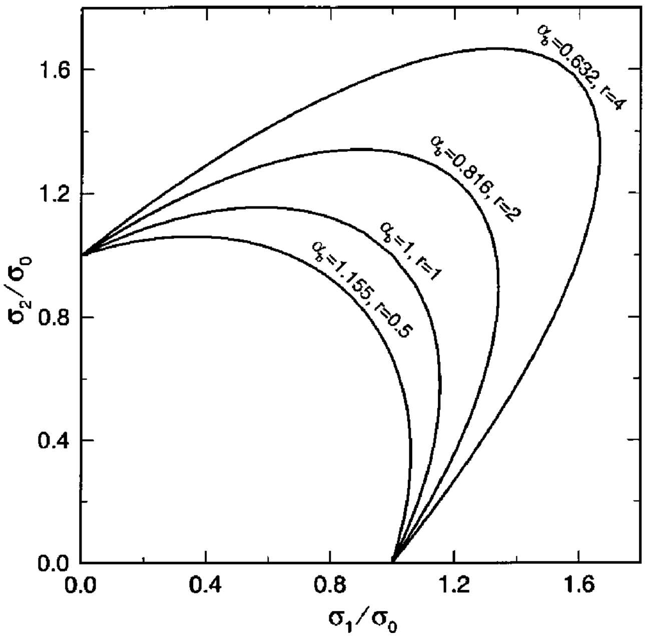

Fig. 1 The loci of Hill's 1948, 1979, and 1993 criteria coincide under the condition of Eq 3.

$$\sigma_b = \sqrt{\frac{r+1}{2}}\,\sigma_u \qquad \text{(Eq 3)}$$

Eq 3 indicates that for an r value larger than unity, the biaxial tension yield stress σ_b must be larger than the uniaxial tension yield stress σ_u, or *vice versa.*

2.2 The 1979 Yield Criterion

For aluminum sheet, the r value is normally less than unity (between 0.6 and 0.8). Experimental data show that for most aluminum sheet, the yield stress for balanced biaxial tension is larger than the yield stress for uniaxial tension (anomalous behavior). Therefore, Eq 3 contradicts the physical phenomenon of aluminum sheets. This indicates that the 1948 criterion may encounter problems when predicting forming limits of aluminum. In order to deal with the anomaly, Hill proposed the second yield criterion in 1979:[13]

$$f|\sigma_2-\sigma_3|^M + g|\sigma_3-\sigma_1|^M + h|\sigma_1-\sigma_2|^M + a|2\sigma_1-\sigma_2-\sigma_3|^M$$
$$+b|2\sigma_2-\sigma_3-\sigma_1|^M + c|2\sigma_3-\sigma_1-\sigma_2|^M = \sigma^M \qquad \text{(Eq 4)}$$

There are seven parameters in Eq 4. They are determined by uniaxial tension test in the three orthotropic directions, together with three transverse strain ratios, plus one other combined loading test (such as the biaxial tension test). For in-plane isotropy, the four simple forms of Eq 4 were given by Hill (Ref 13, Appendix 1). Lian *et al.* pointed out that the yield locus of the fourth equation remains convex as long as the exponent M is greater than unity.[16] The present analysis will focus on this equation. Using both uniaxial tension yield stress σ_u and r value, the fourth equation becomes

$$|\sigma_1+\sigma_2|^M + (1+2r)\,|\sigma_1-\sigma_2|^M = 2(1+r)\,\sigma_u^M \qquad \text{(Eq 5)}$$

The effective strain is given by work equivalence as

$$\varepsilon = \frac{1}{2}\{2(1+r)\}^{\frac{1}{M}}$$
$$\left[|\varepsilon_1+\varepsilon_2|^{\frac{M}{M-1}} + (1+2r)^{-\frac{1}{M-1}}|\varepsilon_1-\varepsilon_2|^{\frac{M}{M-1}}\right]^{\frac{(M-1)}{M}} \qquad \text{(Eq 6)}$$

It is noted that when $M = 2$, Eq 5 reduces to the 1948 yield function, Eq 2. Substituting the yield stress under balanced biaxial tension σ_b into Eq 5 results in

$$\left(\frac{\sigma_b}{\sigma_u}\right)^M = \frac{1+r}{2^{M-1}} \qquad \text{(Eq 7)}$$

Equation 7 indicates that the anomalous behavior exists with $2^{M-1} > 1 + r$ when $r > 1$, or $2^{M-1} < 1 + r$ when $r < 1$.[13] Denoting $\alpha_b = \sigma_u/\sigma_b$, the stress exponent M is obtained from Eq 7:

$$M = \frac{\ln(2(1+r))}{\ln(2/\alpha_b)} \qquad \text{(Eq 8)}$$

2.3 The 1993 Yield Criterion

The yield function proposed by Hill in 1993 is [14]

$$\frac{\sigma_1^2}{\sigma_0^2} - \frac{c\sigma_1\sigma_2}{\sigma_0\sigma_{90}} + \frac{\sigma_2^2}{\sigma_{90}^2} + \left\{(p+q) - \frac{p\sigma_1+q\sigma_2}{\sigma_b}\right\}\frac{\sigma_1\sigma_2}{\sigma_0\sigma_{90}} = 1 \qquad \text{(Eq 9)}$$

where c, p, and q are nondimensional parameters given by

$$\frac{c}{\sigma_0\sigma_{90}} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma_{90}^2} - \frac{1}{\sigma_b^2} \qquad \text{(Eq 10)}$$

$$\left(\frac{1}{\sigma_0} + \frac{1}{\sigma_{90}} - \frac{1}{\sigma_b}\right)p = \frac{2r_0(\sigma_b-\sigma_{90})}{(1+r_0)\sigma_0^2} - \frac{2r_{90}\sigma_b}{(1+r_{90})\sigma_{90}^2} + \frac{c}{\sigma_0}$$
$$\left(\frac{1}{\sigma_0} + \frac{1}{\sigma_{90}} - \frac{1}{\sigma_b}\right)q = \frac{2r_{90}(\sigma_b-\sigma_0)}{(1+r_{90})\sigma_{90}^2} - \frac{2r_0\sigma_b}{(1+r_0)\sigma_0^2} + \frac{c}{\sigma_{90}} \qquad \text{(Eq 11)}$$

In the above equations, σ_0 *and* σ_{90} are yield stresses for uniaxial tension at 0° and 90° to the rolling direction, respectively, and r_0 *and* r_{90} are ratios of transverse to through-thickness strain corresponding to σ_0 *and* σ_{90}, respectively.

Similar to the definition of α_b, it is assumed that the ratio of the yield stresses σ_0 *and* σ_{90} also remains constant so that $\sigma_0 = \alpha_0\sigma_{90}$. Then, Eqs 6 to 8 can be written as

$$\frac{\sigma_1^2}{\sigma_0^2} - \alpha_0\frac{c\sigma_1\sigma_2}{\sigma_0^2} + \alpha_0^2\frac{\sigma_2^2}{\sigma_0^2}$$
$$+\alpha_0\left\{(p+q) - \alpha_b\frac{p\sigma_1+q\sigma_2}{\sigma_0}\right\}\frac{\sigma_1\sigma_2}{\sigma_0^2} = 1 \qquad \text{(Eq 12)}$$

where

$$c = \frac{1}{\alpha_0} + \alpha_0 - \frac{\alpha_b^2}{\alpha_0} \qquad \text{(Eq 13)}$$

$$(1+\alpha_0-\alpha_b)p = \frac{2r_0(\alpha_0-\alpha_b)}{(1+r_0)\alpha_0\alpha_b} - \frac{2r_{90}\alpha_0^2}{(1+r_{90})\alpha_b} + c$$
$$(1+\alpha_0-\alpha_b)q = \frac{2r_{90}(1-\alpha_b)\alpha_0^2}{(1+r_{90})\alpha_b} - \frac{2r_0}{(1+r_0)\alpha_b} + \alpha_0 c \quad \text{(Eq 14)}$$

It is noted that if α_0, α_b, r_0, and r_{90} are all selected to be unity, Eq 14 becomes the von Mises yield criterion. It is also interesting to point out that for the case of in-plane isotropy, Hill's 1993 and 1979 yield functions (Eq 12 and 5) reduce to his 1948 yield function (Eq 2) if α_b is determined from Eq 3. The yield locus for this situation is shown in Fig. 1.

Figure 2 compares these three criteria. For the 1979 and 1993 yield criteria, α_b is set to be unity. Since α_b is determined by r in the 1948 yield criterion, the biaxial yield stress σ_b and the shape of the yield locus are completely different from those of the 1979 and 1993 criteria. It is observed that for $r > 1$, the locus of the 1979 yield criterion is more flattened near balanced biaxial tension than that of the 1993 yield criterion, while for $r < 1$, the yield locus of the 1993 criterion is more flattened than that of the 1979 criterion. The yield locus variations of the 1979 and 1993 yield functions with α_b are shown in Fig. 3, where the r value is a constant (= 1). It is observed that for $\alpha_b > 1$ the yield locus of the 1979 yield function is more flattened than that of the 1993 yield locus and *vice versa.* It will be shown later that a slight change in the shape of the yield locus as shown in Fig. 2 and 3 will influence forming limits significantly. It is noted that the constitutive behavior of the material may also be as important as the yield function in affecting the forming limit diagram. However, the current study will mainly concentrate on the yield function effect on FLDs.

3. Method of Analysis for FLDs

Two analytical methods (the M-K method and the bifurcation analysis) are used to compare the yield criteria in predicting forming limits. The M-K method used in the prediction of FLDs from the 1948 and 1979 yield criteria is similar to that proposed by Graf and Hosford.[19] However, for the 1993 yield criterion, the M-K method proposed by Xu and Weinmann[20] is used to predict FLDs. The bifurcation analysis is based on the Hutchinson and Neale approach for sheet metals following the von Mises yield criterion.[7] A more general form of instantaneous moduli in the rate form of the constitutive law has been derived for anisotropic sheet materials (Appendix 2). To account for strain rate sensitivity in the bifurcation analysis, an approximate method is proposed here to calculate the ratio of the tangent modulus to the secant modulus.[21] The constitutive equation is in the form of the power law

$$\sigma = K\varepsilon^n \dot{\varepsilon}^m \quad \text{(Eq 15)}$$

where σ is the effective stress; ϵ and $\dot{\epsilon}$ are effective strain and strain rate; and n and m are the strain hardening exponent and strain rate sensitivity exponent, respectively. The tangent modulus is thus given by

$$E_t = \frac{\dot{\sigma}}{\dot{\varepsilon}} = K\left(n\varepsilon^{n-1}\dot{\varepsilon}^m + m\varepsilon^n\dot{\varepsilon}^{m-1}\ddot{\varepsilon}\right) \quad \text{(Eq 16)}$$

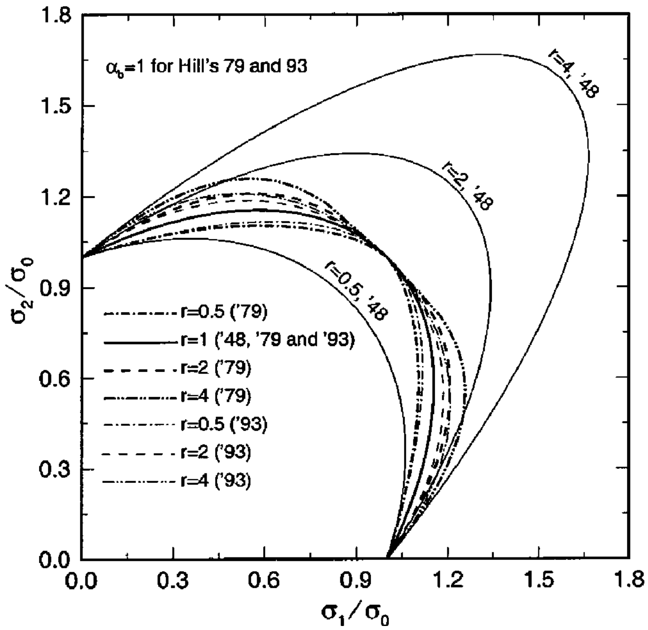

Fig. 2 Comparison of Hill's three yield criteria

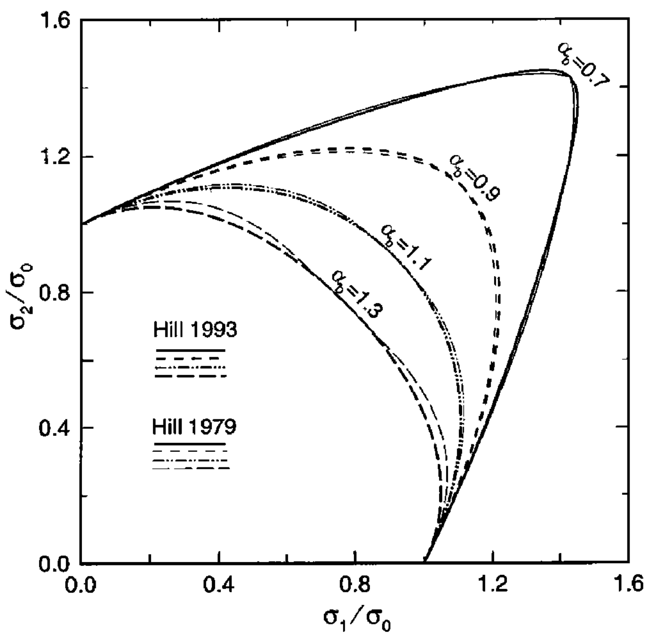

Fig. 3 The variations of loci with α_b for Hill's 1979 and 1993 yield criteria (r = 1)

where $\dot{\sigma}$ and $\dot{\epsilon}$ are the time derivatives of the effective stress and the strain rate, respectively. The secant modulus is defined as

$$E_s = \frac{\sigma}{\varepsilon} = K\varepsilon^{n-1}\dot{\varepsilon}^m \quad \text{(Eq 17)}$$

Therefore, the ratio of the tangent modulus to the secant modulus (which is required in the bifurcation analysis) is

$$\frac{E_t}{E_s} = n + m\frac{e_t}{e_s} \quad \text{(Eq 18)}$$

In Eq 18, e_t and e_s may be called the tangent and the secant moduli of the strain rate and strain curve. They are given by

$$e_t = \frac{d\dot{\varepsilon}}{d\varepsilon},\ and\ e_s = \frac{\dot{\varepsilon}}{\varepsilon}$$

If it is assumed that the dependence of the strain rate on the strain within the neck also follows the power-law relationship, *i.e.*,

$$\dot{\varepsilon} = K_1\varepsilon^c \qquad \text{(Eq 19)}$$

Eq 18 reduces to the simple form

$$\frac{E_t}{E_s} = n + mc \qquad \text{(Eq 20)}$$

where c is the exponent in the strain rate and strain relationship (Eq. 19). It is noted that the strain rate and strain relation in sheet metals may not follow the form of the power law, and it may be influenced by many factors including material properties and loading conditions. Elaborate experiments should be carried out on a case-by-case basis to determine this relationship. Therefore, the above treatment is only an approximation. However, it is observed from Eq. 20 that for a positive c, a positive strain rate sensitivity exponent m will increase the ratio of the tangent modulus to the second modulus, which will increase the forming limit under plane strain condition (FLD_0) and decrease the slope of the forming limit curve in the regime of positive minor strains. [21] This trend agrees with experimental observation. A simple and easy way to determine c is to fit the predicted FLD_0 with experimental data.

4. Comparison of Forming Limits

In order to compare the forming limits predicted from Hill's three criteria, both the M-K method and bifurcation method are used in the analysis. Since α_b in the 1948 criterion is not an independent parameter, forming limits predicted from this criterion are not directly comparable to those predicted from the 1979 and 1993 criteria. Figures 4 and 5 illustrate forming limit predictions from the 1948 criterion, where the M-K predictions are similar to those by Parmer and Mellor[4] and Graf and Hosford.[19] A strong dependence of forming limits on the r value is predicted, although the bifurcation analysis tends to reduce this dependence somewhat. Under biaxial tension, forming limits predicted using the bifurcation analysis are much lower than those from the M-K method due to the fact that the deformation theory allows for vertex formation on the yield surface.[6] However, under plane strain conditions, the bifurcation analysis gives a higher forming limit prediction (FLD_0) than the M-K method, since no imperfection is introduced in the bifurcation analysis. It is noted that the deformation localization process is also dependent on the strain-rate sensitivity of the material. A high strain rate hardening exponent m will help the material balance the effect of geometrical defects and retard the localization process, and thus increase formability. The low-carbon steels normally have moderately high strain-rate hardening exponents, while aluminum alloys usually exhibit strain-rate insensitivity or even

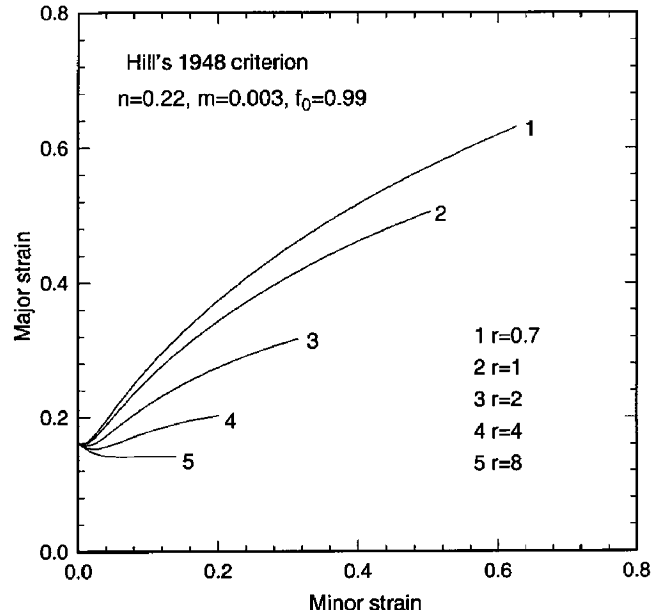

Fig. 4 Effect of r value on forming limits predicted using the M-K method and Hill's 1948 yield criterion

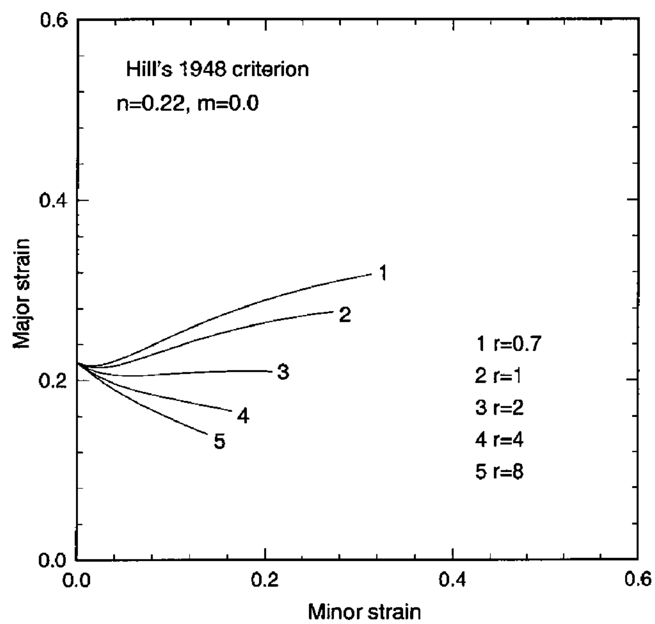

Fig.5 Effect of r value on forming limits predicted using the bifurcation analysis and Hill's 1948 yield criterion

strain-rate softening (negative m value).[22] To concentrate on the effect of the yield locus on FLDs, however, a very small positive strain rate exponent ($m = 0.003$) is used for the M-K analysis and $m = 0$ is used for the bifurcation analysis in Fig. 4 to 13, which leads to FLD_0 values equal to and less than the n value for the bifurcation and the M-K analyses, respectively. The strain rate effect will be taken into consideration in the comparison of predicted FLDs with experimental data in Fig. 14 to 17.

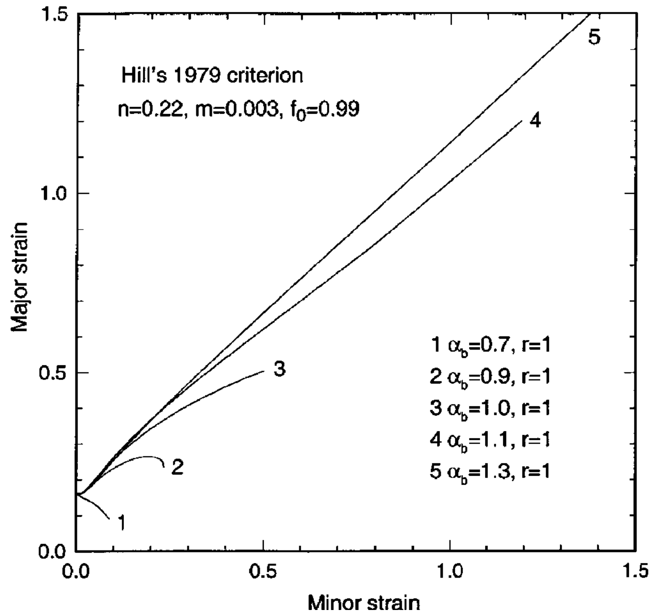

Fig. 6 Effect of α_b on forming limits predicted using the M-K method and Hill's 1979 yield criterion

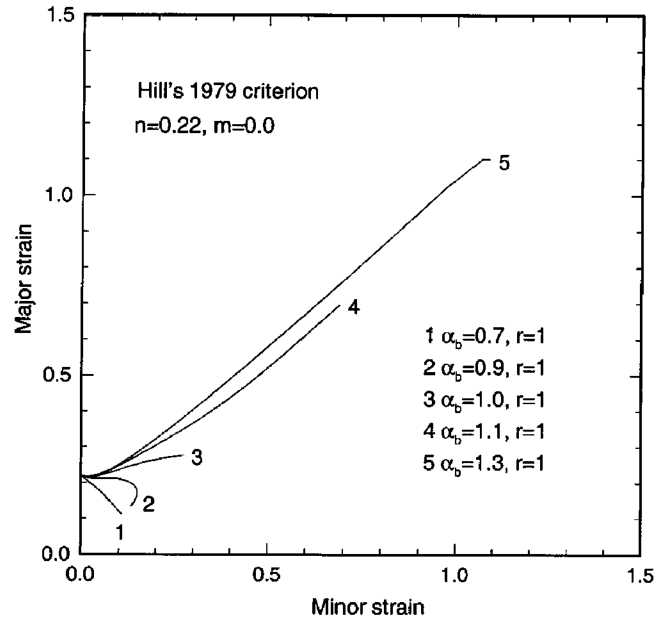

Fig. 8 Effect of α_b on forming limits predicted using the bifurcation analysis and Hill's 1979 yield criterion

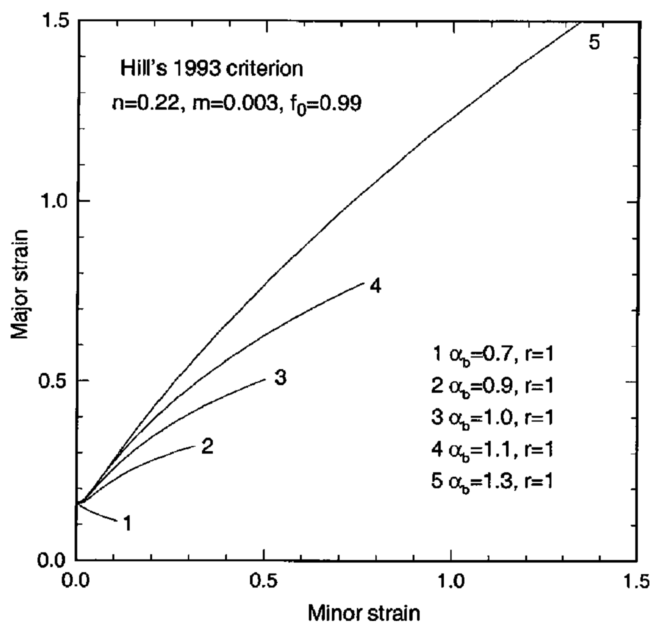

Fig. 7 Effect of α_b on forming limits predicted using the M-K method and Hill's 1993 yield criterion

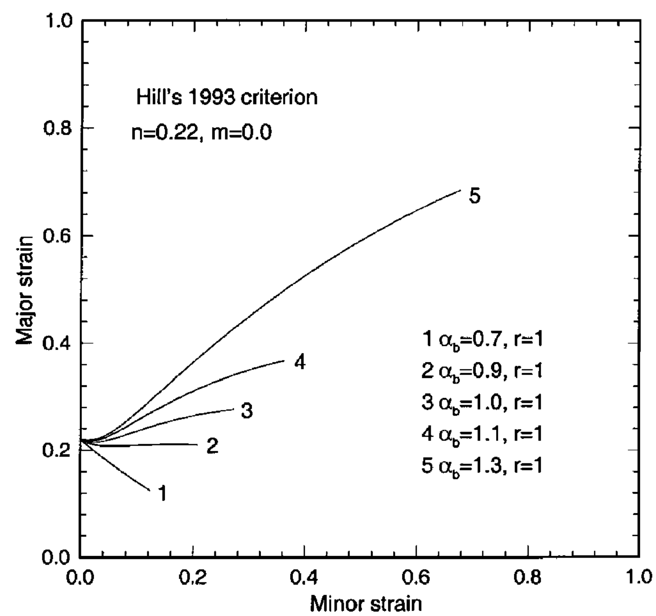

Fig. 9 Effect of α_b on forming limits predicted using the bifurcation analysis and Hill's 1993 yield criterion

To compare the effect of the 1979 and 1993 yield criteria on forming limits, α_b in the 1979 criterion is selected to be the same as that in the 1993 criterion for each case. The stress exponent M in the 1979 criterion is calculated using Eq. 8. The effects of α_b on forming limits predicted using the M-K method are shown in Fig. 6 and 7 for the 1979 and 1993 yield criteria, respectively. The range of α_b in the figures is selected in such a way that a wide spectrum of sheet materials can be reflected in the simulation. It is observed that for $\alpha_b < 1$ forming limits near balanced biaxial tension predicted from the 1993 yield criterion are higher than those from the 1979 yield criterion, while for $\alpha_b > 1$, forming limits near balanced biaxial tension predicted from the 1979 are higher than those predicted from the 1993 yield criterion. This is due to the fact that for $\alpha_b > 1$ the shape of the 1979 yield locus near balanced biaxial tension is much more flattened than that of the 1993 yield locus, and *vice versa.* The forming limits predicted using the bifurcation analysis are shown in Fig. 8 and 9 for the 1979 and 1993 yield criteria.

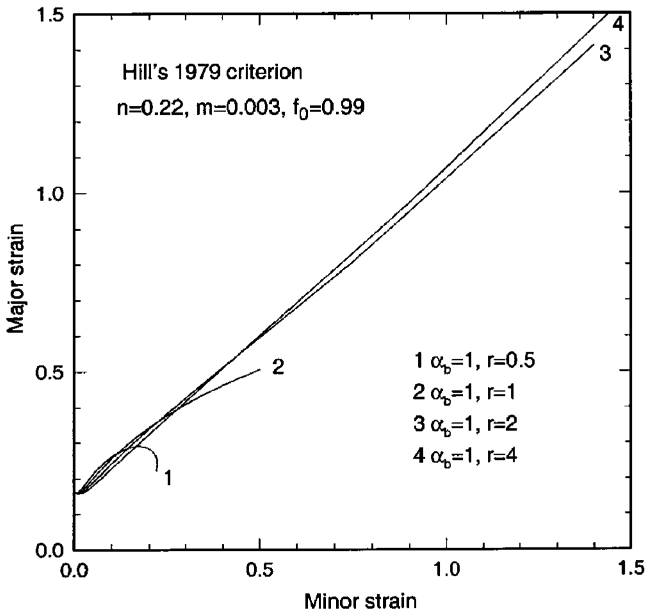

Fig. 10 Effect of the *r* value on forming limits predicted using the M-K method and Hill's 1979 yield criterion

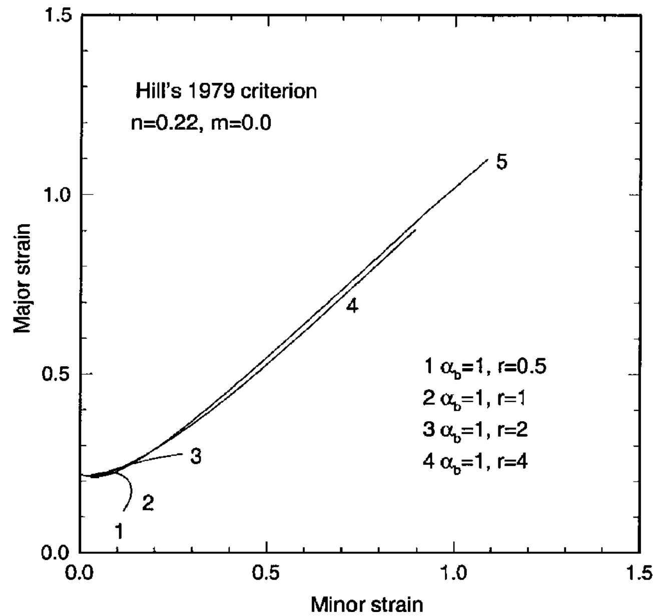

Fig. 12 Effect of the *r* value on forming limits predicted using the bifurcation analysis and Hill's 1979 yield criterion

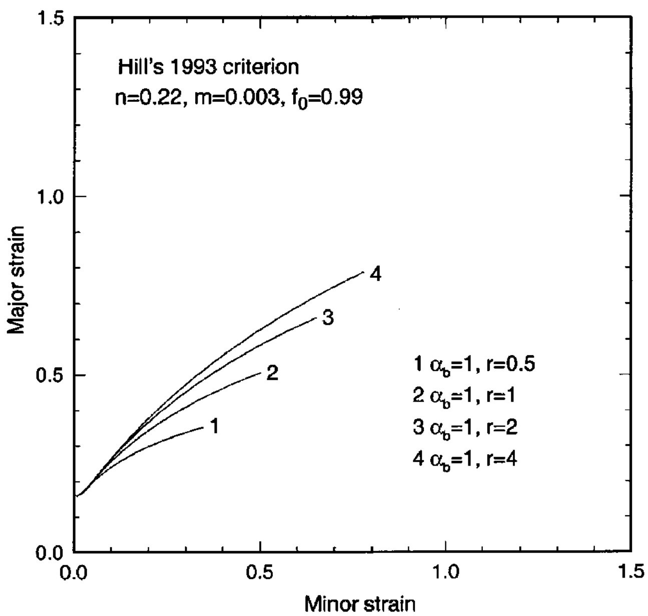

Fig. 11 Effect of the *r* value on forming limits predicted using the M-K method and Hill's 1993 yield criterion

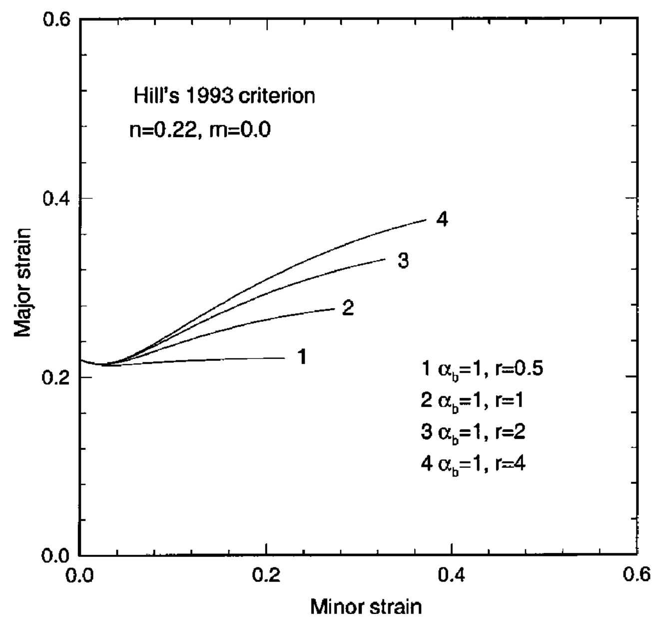

Fig. 13 Effect of the *r* value on the forming limits predicted using the bifurcation analysis and Hill's 1993 yield criterion

A trend similar to that in Fig. 6 and 7 is observed. However, forming limits predicted from the bifurcation analysis are much lower under balanced biaxial tension than those predicted from the M-K method, and the forming limit curves become concave upward for α_b >1 for the 1979 yield criterion in the bifurcation analysis, which seems contradictory to most of the experimental observations.

The effects of the *r* value on forming limits predicted using the M-K method are shown in Fig. 10 and 11, while Fig. 12 and 13 are predictions of the bifurcation analysis. It is observed that for $r > 1$, forming limits predicted using the 1979 yield criterion are higher than those predicted using the 1993 yield criterion, since a significant difference in the curvature of the yield locus exists near balanced biaxial tension between the 1979 and 1993 yield functions (Fig. 2). For $r = 0.5$, the sudden drop in forming limits near balanced biaxial tension in Fig. 10 and 12 indicates a significant increase in the curvature of the 1979 yield locus at balanced biaxial tension. Comparison between Fig. 10 and 11

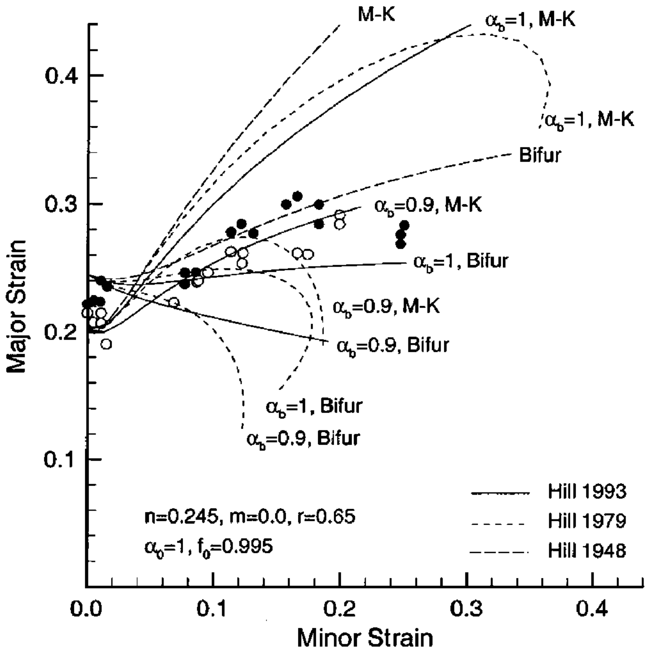

Fig. 14 Comparison of predicted forming limits with experimental data obtained from aluminum 6111-T4. The dots and circles represents measured strains on neck-affected and neck-free strains circles.[22]

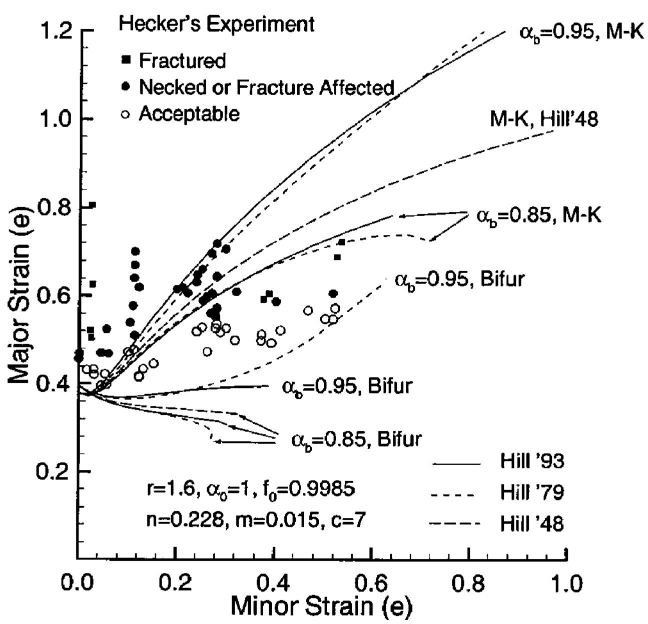

Fig. 15 Comparison of predicted forming limits with experimental data obtained from AK steel.[24]

and between Fig. 12 and 13 indicates that the r value has less effect on forming limits predicted using the 1993 yield criterion than those predicted using the 1979 yield criterion. Considering that for most sheet metal materials the r value has an insignificant influence on the FLDs [19] and α_b changes only slightly with the r value, Hill's 1993 criterion seems to be superior to his 1979 criterion in predicting forming limits of sheet metals.

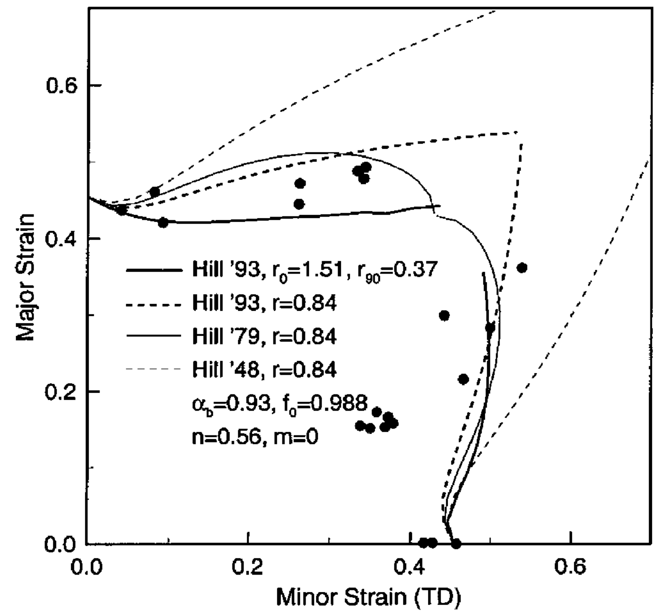

Fig. 16 Comparison of predictions from the M-K method with experimental data obtained on Brass70/30 thin wall tube[25]

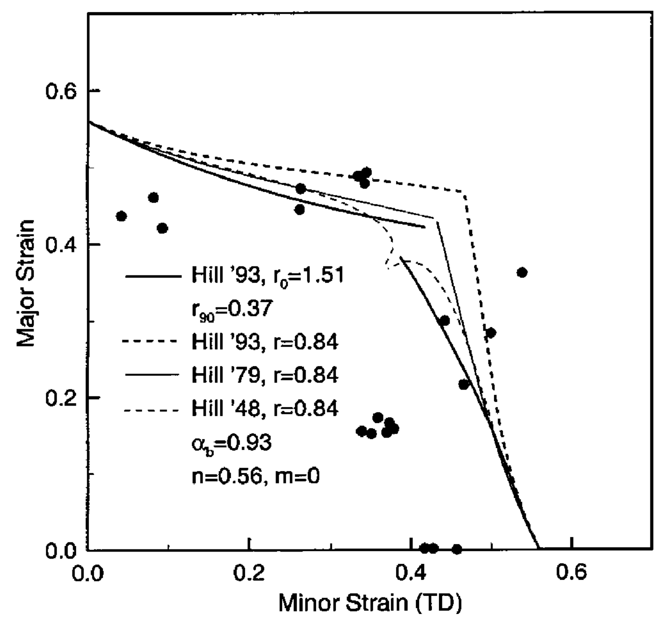

Fig. 17 Comparison of predictions from the bifurcation analysis with experimental data obtained on brass 70/30 thin wall tube[25]

Figure 14 shows the comparison of predicted forming limits with experimental data obtained for aluminum 6111-T4.[22] Experimental data for α_b is not available for this particular material. However, data for other similar materials[18] indicate that α_b for aluminum seems very close to 0.9, although measured data vary considerably.[23] For the purpose of comparison, predictions for $\alpha_b = 1$ are also illustrated to show the sensitivity of FLDs to these parameters. It is observed that the bifurcation analysis in conjunction with the 1979 yield criterion provides predictions that are much lower than the experimental data near balanced biaxial tension. With $\alpha_b = 1.0$ and the M-K method, the 1979 and 1993 criteria yield forming limit predictions much higher than experimental data, indicating that the analysis is very sensitive to α_b. The highest forming limit curve is predicted from the 1948 yield criterion and the M-K analysis. Predictions based on the 1993 criterion, the M-K analysis, and the experimentally suggested α_b (= 0.9) are observed to be in good agreement with experimental data. Given the scatter in the experimental data, it can also be shown that the bifurcation analysis based on the 1948 criterion yields acceptable predictions of FLDs, although they are slightly higher than experimental data. Although a small negative value of *m* was measured in experiments,[22] the rate sensitivity is not considered in the analysis.

Figure 15 shows the comparison of predicted FLDs with experimental data for AK steel.[24] A value of 7 for *c* is used in the bifurcation analysis to fit the predicted FLD_0 with the experimental data. For steels with *r* value around 1.6, α_b was found to be close to 0.85.[23] However, FLDs for $\alpha_b = 0.95$ are also presented for comparison. It is shown that for the three yield criteria, the bifurcation analysis provides FLDs much lower than experimental data. For the 1948, and the 1979 and 1993 yield criteria with $\alpha_b = 0.85$, forming limit predictions based on the M-K method are shown to agree reasonably with experimental observations, although they are slightly higher. However, when $\alpha_b = 0.95$ is used, forming limit predictions based on the M-K method are significantly higher than experimental data, showing that α_b is a critical parameter in predicting forming limits based on Hill's 1979 and 1993 criteria, and it should be determined by carrying out uniaxial and biaxial tension tests carefully.

Brass is another commonly used material in sheet metal forming. Figure 16 illustrates the comparison of experimental data with forming limits predicted using the M-K method,[25] where α_b (0.93) is determined based on experimental observations.[18, 26, 27] It is shown that only the 1993 yield criterion provides a prediction that follows the trend of the experimental data approximately due to the fact that the 1993 yield criterion is designed particularly for this type of material. It is noted that the experiment was conducted on thin tubes with the wall thickness of 0.508 mm rather than flat sheets. This may contribute to the discrepancy between analytical and experimental results.[25] Figure 17 shows the comparison of the same experimental data as in Fig. 16 with forming limits predicted based on the bifurcation analysis. The difference among the results predicted using all three yield criteria is relatively small compared with that based on the M-K analysis. However, the predictions are unsatisfactory.

5. Conclusions

The yield function and the analytical method used have a significant effect on the predicted FLDs. The bifurcation analysis based on the deformation theory of plasticity provides forming limit predictions lower than those of the M-K approach based on the flow theory of plasticity. For aluminum alloys, the M-K analysis in conjunction with Hill's 1993 yield criterion provides forming limit predictions in good agreement with experimental data. For AK steel, the M-K analysis based on all three yield criteria provides a reasonable prediction of forming limits (though slightly higher). For brass, only the M-K analysis based on the 1993 yield criterion can provide forming limit predictions close to the trend of experimental data.

It is noted that Hill's 1993 yield function is defined in the principal stress space, which excludes the shear stresses. This means that this yield function can only be used for the right-hand side of the FLD analysis. However, Hill's analysis shows that the deformation localization of thin sheets occurs along the zero extension direction in the negative minor strain regime.[28] This indicates that the left-hand side of the FLD should not depend on the yield function based on the flow theory of plasticity, as shown in several special cases,[29,30] and, therefore, this portion of the FLD should be a straight line along the constant thinning direction. For stretching operations, the above comparison shows that the combination of Hill's 1993 yield criterion and the M-K method provides reasonable forming limit predictions for a wide range of materials, which indicates that Hill's 1993 yield function seems superior to his 1948 and 1979 yield functions in FLD predictions.

Since there are no shear stress terms in Hill's 1993 and 1979 yield functions, it is not appropriate to implement them in any finite element code. Hill's 1948 yield function has a simple and quadratic form and is defined in the complete stress space. Therefore, it is widely used in finite element analyses. However, this yield function is suitable to steel only. For aluminum, the 1948 yield function may lead to inaccurate results in the finite element simulation,[31] since it cannot characterize the anomalous behavior of aluminum. To analyze aluminum forming correctly, the finite element code has to incorporate a yield function, which is defined in the complete stress space and which can represent the anomalous behavior of aluminum as those proposed by Barlat *et al.*[11, 32]

Appendix 1

The four simple forms of Hill's 1979 yield criterion (Eq 4) under planar isotropic condition are as follows:

(1) $a = b = 0, f = g, h = 0$
(2) $a = b, c = 0, f = g = 0$
(3) $a = b, c = 0, f = g, h = 0$
(4) $a = b = 0, f = g = 0$

The corresponding forms of yield functions for plane stress condition are as follows:

$$(1)\ c|\sigma_1 + \sigma_2|^M + f\left(|\sigma_1|^M + |\sigma_2|^M\right) = \sigma^M$$
$$(2)\ a|2\sigma_1 - \sigma_2|^M + f\,2\sigma_2 - \sigma_1|^M + h|\sigma_1 - \sigma_2|^M = \sigma^M$$
$$(3)\ a\left(|2\sigma_1 - \sigma_2|^M + |2\sigma_2 - \sigma_1|^M\right) + f\left(|\sigma_1|^M + \sigma_2|^M\right) = \sigma^M$$
$$(4)\ c|\sigma_1 + \sigma_2|^M + h|\sigma_1 - \sigma_2|^M = \sigma^M$$

Appendix 2

The governing equation in the bifurcation analysis for material with an arbitrary yield function is

$$\hat{L}_{11} = \sigma_1 \qquad \text{(Eq A2 – 1)}$$

where $\hat{L}_{11}$ is one of the instantaneous moduli for the rate form constitutive equation:

$$\begin{aligned} \tilde{\sigma}_1 &= \hat{L}_{12}\dot{\varepsilon}_{11} + \hat{L}_{22}\dot{\varepsilon}_{22} \\ \tilde{\sigma}_2 &= \hat{L}_{12}\dot{\varepsilon}_{11} + \hat{L}_{22}\dot{\varepsilon}_{22} \end{aligned} \qquad \text{(Eq A2 – 2)}$$

The moduli are given by

$$\begin{aligned} \hat{L}_{11} &= \frac{a_{11}b_{22} - a_{21}b_{12}}{b_{11}b_{22} - b_{21}b_{12}} \\ \hat{L}_{12} &= \frac{a_{12}b_{22} - a_{22}b_{12}}{b_{11}b_{22} - b_{21}b_{12}} \\ \hat{L}_{22} &= \frac{a_{11}b_{21} - a_{21}b_{11}}{-b_{11}b_{22} + b_{21}b_{12}} \end{aligned} \qquad \text{(Eq A2 – 3)}$$

and

$$\begin{aligned} a_{11} &= (E_t + E_s)\varepsilon_e - f_1\sigma_1 \\ a_{12} &= -f_1\sigma_2 \\ a_{21} &= -f_2\sigma_1 \\ a_{22} &= (E_t + E_s)\varepsilon_e - f_2\sigma_2 \\ b_{11} &= f_1\varepsilon_1 + (E_t + E_s)\varepsilon_e^2\left(\frac{\partial f_1}{\partial\sigma_1} + \frac{\partial f_1}{\partial\sigma_e} f_1\right) \\ b_{12} &= f_1\varepsilon_2 + (E_t + E_s)\varepsilon_e^2\left(\frac{\partial f_1}{\partial\sigma_2} + \frac{\partial f_1}{\partial\sigma_e} f_2\right) \\ b_{21} &= f_2\varepsilon_1 + (E_t + E_s)\varepsilon_e^2\left(\frac{\partial f_2}{\partial\sigma_1} + \frac{\partial f_2}{\partial\sigma_e} f_1\right) \\ b_{22} &= f_2\varepsilon_2 + (E_t + E_s)\varepsilon_e^2\left(\frac{\partial f_2}{\partial\sigma_2} + \frac{\partial f_2}{\partial\sigma_e} f_2\right) \end{aligned} \qquad \text{(Eq A2 – 4)}$$

In the above equations, a is the strain ratio and f_i is given by

$$f_i = \frac{\partial\sigma_e}{\partial\sigma_i} \qquad \text{(Eq A2 – 4)}$$

From Eq. A2-1 limit strains with different strain ratios are determined.

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